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Optimal Control of the Cascaded Kainji-Jebba Hydroelectric Power Station Based on Conjugate Gradient

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Abstract: Electricity supply in Nigeria has been far below the estimated demand and the installed capacity of the plants. Hence, there has been continuous research on improving the performance of the existing plants, but the government mainly focuses on increasing the installed capacity. This paper presents the determination of the optimal release of water to maximize the energy generation potential of the cascaded Kainji-Jebba hydroelectric power station in Nigeria. The problem was formulated as an optimal control problem with an objective of minimizing the deviation of the head of the Jebba reservoir within a set limit. A conjugate gradient algorithm was then used as a direct solution to the optimal control problem. The computed control law and the resulting state trajectories of 2% error affirms the solution to be genuine and reliable. The algorithm is recommended for use in the design of a real time optimal controller for the system and a decision guide for the operators.

Keywords: Conjugate gradient, Inflow, Operating head, Optimal control, Performance index.

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1. INTRODUCTION

The Cascaded Kainji Hydroelectric Power station (KHEPS) and Jebba Hydroelectric Power Station (JHEPS) are pivot to the Nigeria energy generation [1, 2]. The two stations contribute approximately 17.5% to the installed capacity and the generation capacity has been around 18% on the average. Both take their sources of energy from the River Niger with 103 km separating their dams. Since their prime mover is renewable, the energy generation is relatively cheaper than those of the thermal plants [3].

KHEPS is designed with variable blade Kaplan turbo-alternators with the operating head varying between $24\ m$ to$42\ m$. JHEPS has fixed blade propeller turbine operating at relatively constant head and all units operate at base load. Based on this system design, the operating head of JHEPS is highly sensitive to water release from KHEPS and the number of operating units. The amount of water that must be released from KHEPS and the duration of the release such as to maintain JHEPS reservoir at relatively constant head is the problem yet unsolved [4 – 6].

The two power stations operate in cascade but lack a control system regulating their operation. They are being managed by experience and intuition of the operators [7]. From the operational report, there are occasions where some units at JHEPS are shutdown if the release from KHEPS is low. Sometimes the release from KHEPS is so large that they are spilt at JHEPS reservoir due to unit's failure. There are also occasions where JHEPS can only operate few units and instead of KHEPS to run all its available units, they allow the reservoir head to rise while the water is also evaporated. It is obvious that the operational procedure based on intuition and experience can leads to in poor utilization of available energy resources and lower power generation [1].

Based on the available information gathered, the problem of real-time optimal management of resources between KHEPS and JHEPS has not been solved. Despite all the research efforts made so far, operators still rely on intuitive water release rules to maximise power generation regularly. There are no proper scientifically motivated techniques for the management of resources between the two stations. Most methods being proposed are mere parameter optimization technique which are only meant for optimizing the static and not dynamic system. The system dynamics is also nonlinear, hence most linear techniques may not be applicable [3 – 6].

Therefore, the problem is purely an optimal control problem where by a control signal is desired that will force the reservoir head at JHEPS to move from an initial point to the desired point in finite time and subject to constraints imposed by the system dynamics. Unfortunately, many problems that are rooted in nonlinear optimal control theory do not have computable solutions or they have solutions that may be obtained only with a great deal of computing effort [8]. Hence this research work considered the proper formulation of the optimal control problem and provision of a suitable technique for determining the optimal control law.

2. DIRECT OPTIMAL CONTROL USING A CONJUGATE GRADIENT ALGORITHM

The methods of solving optimal control problem can be broadly classified into two: the indirect and the direct method. The indirect method applies calculus of variation to set up necessary conditions that must be satisfied by the optimal control. These conditions produce optimal control canonical equations such that their solution ensures that an optimum point has been reached. While using this approach, it is usually necessary to calculate the Hamiltonian, co-state equations, the optimality and transversely conditions [9 – 11].

The direct methods involve the discretization of the state and the control in such a way that the problem is converted to a nonlinear optimization problem or nonlinear programming problem [12, 13]. It starts by subdividing the time interval into $N > 1$ control state: $t_0 < t_1 < t_2 < \cdots < t_N = t_f$. In each of the sub-interval, (t_{i-1}, t_i) approximate $u(t) = u_i(t)$, such that $U(t) = [u_1, u_2, u_3, \ldots\ldots u_N]$. The problem can then be solved using an appropriate nonlinear optimization method.

The optimization technique depends on the problem being solved. Popular techniques can be classified into two: the gradient based, and the heuristic based. There has been no perfect method as each has its own advantages and disadvantages. A designer must study these methods to know the appropriate one for a given control problem. The heuristic based are not widely applicable by engineers as the gradient based [14]. Among the popular gradient based are the steepest descent and the conjugate gradient methods, these two are presented and compared in this work.

In this work, the use of conjugate gradient algorithm to solving the direct optimal control was considered. Since the algorithm is based on the steepest descent, it is believed that it will always yield an optimal control with reduced convergence time. It should be noted that there are slight modifications to the basic algorithm, among is the formulation of the problem such that the performance index is differentiable and the method of computing the optimal step size at each iteration.

Conjugate gradient algorithm is a minimization technique for function of several variable and for finding solution to a simultaneous nonlinear algebraic equation. The method was first applied to solve an unconstrained optimal control problem by [15]. Later, the method was adapted to solve constraint optimal control problem where a penalty function is imposed on the constraints on state and control [16]. It incorporates the fast characteristics of steepest descent method in moving to the optimum but with additional modification in ensuring quick convergence around the optimum.

The procedure starts by guessing and initial control vector $U^{(k)}$ and determine the performance index $J(U^{(k)})$ with the iteration number $k = 0$. Calculate the gradient vector $g^{(k)} = \nabla J(U^{(k)})$ and determine the conjugate gradient parameter $\beta^{(k)}$, where

$$\beta^{(k)} = \frac{\int_{t_0}^{t_f} (g^{(k)})^2 \, dt}{\int_{t_0}^{t_f} (g^{(k-1)})^2 \, dt} \tag{1}$$

$$g^{(k-1)} = 1 \text{ when k} = 0$$

Calculate the direction of search vector s^k.

$$s^k = -g^k + \beta^k s^{k-1} \tag{2}$$

$$s^{(k-1)} = 0 \; when \; k = 0$$

Determine the next control vector $U^{(k+1)}(t)$, where

$$U^{(k+1)}(t) = U^{(k)}(t) + \psi_{min} \, s^k \tag{3}$$

The optimum step size ψ_{min} was computed by performing a minimization search on $J(\, U^{(k)} + \psi \, s^k)$.

Check whether $|J(U^{k+1}) - J(U^k)| \leq 10^{-n}$ to stop the iteration, where n is a constant. If the stopping criterion is not met, increment k by one and repeat the procedure using $U^{(k+1)}$ as the new guess

2.1 Numerical Solution of the Optimal Control using a Conjugate Gradient Algorithm

Optimal control problem requires the system model equation, the performance index and the associated constraints. The system dynamical model is the JHEPS nonlinear operating head dynamics described by Eq. 4 to 7. [1]

$$\frac{dh(t)}{dt} = -\,n\frac{A_2}{A_1}(\sqrt{2g})h^{1/2} + \frac{1}{A_1}(Q_J - Q_L - Q_s) \tag{4}$$

$$Q_J = q_k + Q_{sk} + Q_{CJ} \tag{5}$$

$$u(t) = Q_J(t) - Q_L(t) - Q_s(t) \tag{6}$$

where t represents time (s), h is the operating head (m) and the state variable, u is the control signal, n represents the number of operating units (integer number 1 to 6), A_1 is the effective surface area of the reservoir, A_2 is the effective area of the scroll casing, g represents the acceleration due to gravity, Q_J is the inflow into JHEPS, Q_L is the evaporation loss on JHEPS, Q_s is the spillway discharge from JHEPS, q_k represents the total discharge from KHEPS tailrace, Q_{sk} represents the spillway discharge from KHEPS, Q_{CJ} is the inflow from catchment area between KHEPS and JHEPS and f represents a nonlinear function.

Eq. 1 can be written in the standard form as Eq. 7;

$$\dot{h}(t) = \boldsymbol{f}(h(t), u(t), \, t) \;\; ; \; t_0 \leq t \leq t_f \tag{7}$$

The optimal control problem is the determination of the control signal $u(t)$ to be released from KHEPS (actuator) that will force the operating head $h(t)$ of JHEPS to move from an initial point $h(t_0)$ to a final point $h(t_f)$ within a given time $(t_0 \rightarrow t_f)$.

Performance Indices ($\boldsymbol{J}$)

The Performance Index (J) was selected to accommodate and appropriately penalizes deviation from a specified head and ensures that the control is bounded. This consist of the integral of the square error from the desired

operational head as can be seen in Eq. 8. The performance index is subject to the system model of Eq. 7 and the boundary conditions of Eq. 9 to 11.
Hence,

$$J(h,u,t) = min \int_{t_0}^{t_f} \left\{K_h\big(h(t) - h(T)\big)^2\right\} dt \qquad (8)$$

Subject to the system constraints:

$$\dot{h}(t) = \boldsymbol{f}(h(t), u(t),\ t)\ \ ;\ t_0 \le t \le t_f$$

$$h(t_0) = \ h_0 \qquad (9)$$

$$h(t_f) = h(T) \qquad (10)$$

Nonlinear penalties on U: $[U_{min}(t),\ U_{max}(t)]$ (11)

where $h(T)$ represents the desired final value for the state and K_h is a positive weighing scalar constant.

The realization optimal control for cascaded Kainji-Jebba hydroelectric power station was carried out in Microsoft EXCEL VBA® programming environment and the procedure are presented below.

Set Up: Let the control vector $U^{(k)}(t)$ be partitioned within the time interval $[t_0, t_f]$, where;

$$\pi[t_0, t_f] = t_0 < t_1 < t_2 < t_3 < t_f = t_4 \text{ ; and}$$

$$U^{(k)}(t) = [u_1^{(k)}(t), u_2^{(k)}(t), u_3^{(k)}(t), u_4^{(k)}(t)]^T$$

Step 1: Let $k = 0$.
Set the initial condition $h^{(k=0)}(t) = h_0$
Guess values for $U^{(k=0)}(t)$ from $t_0 \to\ t_f$.

Step 2: Numerically solve the nonlinear differential equation $\boldsymbol{f}(h(t), u(t), t)$ from $t_0 \to t_f$ to obtain $h^{(k)}(t)$. The Adams – Moulton technique with Adams–Bashforth as predictor and Runge-Kutta for starting was used.

Step 3: Compute the performance index $\boldsymbol{J}^{(k)} = \boldsymbol{J}(u_1^{(k)}, u_2^{(k)}, u_3^{(k)}, u_4^{(k)})$. The Trapezoidal rule was used at this stage.

Step 4: Choose a perturbation value Δu and compute $\boldsymbol{J}_m^{(k)}$, m=1, 2, 3, 4.

$$J_1^{(k)} = J(u_1^{(k)} + \Delta u,\ u_2^{(k)}, u_3^{(k)}, u_4^{(k)})$$

$$J_2^{(k)} = J(u_1^{(k)},\ u_2^{(k)} + \Delta u, u_3^{(k)}, u_4^{(k)})$$

$$J_3^{(k)} = J(u_1^{(k)},\ u_2^{(k)}, u_3^{(k)} + \Delta u, u_4^{(k)})$$

$$J_4^{(k)} = J(u_1^{(k)},\ u_2^{(k)}, u_3^{(k)}, u_4^{(k)} + \Delta u)$$

Step 4: Compute the gradient vector $\boldsymbol{g}_m^{(k)} = \nabla \boldsymbol{J}_m^{(k)}$,

where $\boldsymbol{g}^{(k)} = [g_1^{(k)}, g_2^{(k)}, g_3^{(k)}, g_4^{(k)}\ ]^T$ and

$$g_m^{(k)} = \frac{J^{(k)} - J_m^{(k)}}{\Delta u};\ m = 1, 2, 3, 4$$

Step 5: Compute the conjugate gradient parameter;

$$\beta^{(k)} = \frac{\int_{t_0}^{t_f} (g^{(k)})^2\, dt}{\int_{t_0}^{t_f} (g^{(k-1)})^2\, dt},$$

where $\int_{t_0}^{t_f} (g^{(0)})^2\, dt = 1$

Step 6: Compute the direction of search $s^{(k)} = -g^{(k)} + \beta^{(k)} s^{(k-1)};\ where\ \ s^{(-1)} = 0$

Step 7: Determine $\psi_{min}^{(k)}$ by selecting a set of three Fibonacci numbers ψ_1, ψ_2 and ψ_3, such that

$$U_{\psi_1}^{(k+1)} = U^{(k)} +\ \psi_1^{(k)} s^{(k)}\ and\ \ J_{\psi_1}^{(k)}$$

$$U_{\psi_2}^{(k+1)} = U^{(k)} +\ \psi_2^{(k)} s^{(k)}\ and\ \ J_{\psi_2}^{(k)}$$

$$U_{\psi_3}^{(k+1)} = U^{(k)} +\ \psi_3^{(k)} s^{(k)}\ and\ \ J_{\psi_3}^{(k)}$$

Step 8: Solve for the constants $\boldsymbol{b}$ and $\boldsymbol{c}$

$$\boldsymbol{c} = \frac{1}{(\psi_1 - \psi_3)} \left[\frac{\big(J_{(\psi_1)} - J_{(\psi_2)}\big)}{(\psi_1 - \psi_2)} - \frac{\big(J_{(\psi_2)} - J_{(\psi_3)}\big)}{(\psi_2 - \psi_3)}\right]$$

$$\boldsymbol{b} = \frac{(J_{(\psi_1)} - J_{(\psi_2)})}{(\psi_1 - \psi_3)} - \frac{(\psi_1 + \psi_2)}{(\psi_1 - \psi_3)} \left[\frac{(J_{(\psi_1)} - J_{(\psi_2)})}{(\psi_1 - \psi_2)} - \frac{(J_{(\psi_2)} - J_{(\psi_3)})}{(\psi_2 - \psi_3)}\right]$$

Step 9: Compute $\psi^{(k)}{}_{(min)} = \psi^{(k)} = \ -\frac{\boldsymbol{b}}{\boldsymbol{2c}}$

Step 10: Compute the control vector.

$$\left[U^{(k+1)}\right]^T = \left[U^{(k)}\right]^T +\ \psi_{(min)}^{(k)} \left[s^{(K)}\right]^T$$

Step 11: Check if $\left\|J(u^{(k+1)}\right\| - \left\|J(u^{(k)}\right\| \le 10^{-n}$ and $\frac{\partial J^*}{\partial U^*} \approx 0$, n is a positive constant

If this is true, then $U^*(t) = U^{(k+1)}(t)$ and output $h^*(t)$,
else,
let $k = k + 1$,
$U^{(k)}(t) = U^{(k+1)}(t)$ and return to step 2
End

3. RESULTS AND DISCUSSION

The results obtained from computation of optimal control by a conjugate gradient method are presented in this section. The cases show the realization performed to study the potential of conjugate gradient algorithm in solving a direct optimal control problem and the performance of the technique to ascertain that the solution is optimal. A notation is defined for specifying the operating conditions under a case being considered, the format is as follows: (*Number of Machines*, *Starting head (m)*, *desired final head*, *Number of Days*, *penalty*,). In all computations, a stopping criterial of 10^{-5} was used.

Computation 1: (5, 25.8, 26.1, 1, NOT PENALIZED)

Computation 1 described a situation where five (5) numbers of Turbo-alternator units are running with an operating head at 25.8. It is desired that the operating head moves to 26.1 in 24h, the optimal control is the amount of inflow required to achieve this. The maximum inflow is

Not Penalized. Figure 1 presents the changes in the control after each iteration until the last iterations. The algorithm suggests a control with high inflow for the first 6 hours and decreases the inflow to the minimum in the next six hours. The inflow in the third and last six hours can then be relatively constant and moderate. This is a scientifically motivated policy which is contrary to starting with a low inflow and ending with a moderate one. Hence, Fig. 1 justifies the superiority of scientifically motivated control law as to intuition or experience. The optimization is a minimization problem; hence the performance index is expected to decrease after each iteration. Fig. 2 confirm this postulate, if the performance index does not decrease with iterations, the computation may be wrong, and the final control cannot be optimal.

Two parameters that are peculiar to the control algorithm is the norm of the gradient after each iteration and the optimum step size (λ). The conjugate gradient algorithm necessitate that these two parameters must decrease after each iteration. Fig. 3 and Fig. 4 confirm these two necessities to show that the optimal solution is acceptable.

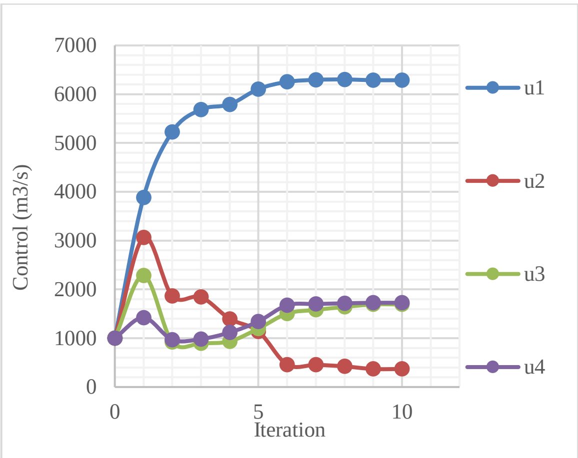

Figure 1. Control vs Iteration for (5, 25.8, 26.1, 1, NOT PENALIZED)

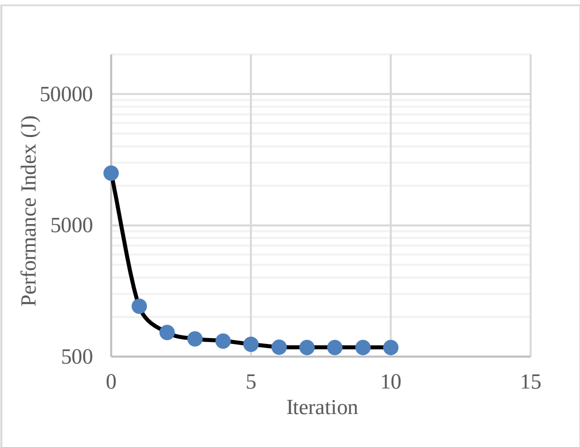

Figure 2. Performance Index vs Iteration for (5, 25.8, 26.1, 1, NOT PENALIZED)

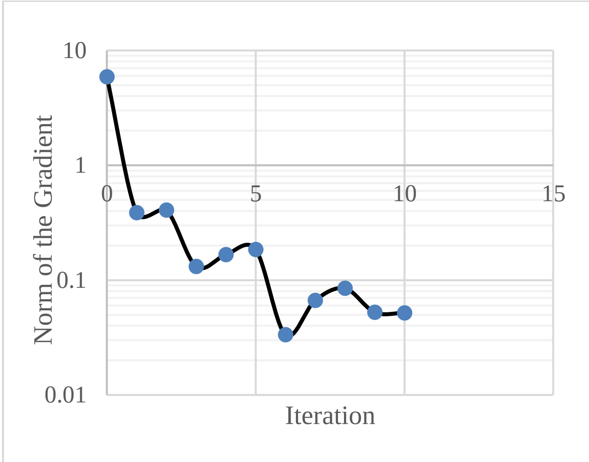

Figure 3. Gradient vs Iteration for (5, 25.8, 26.1, 1, NOT PENALIZED)

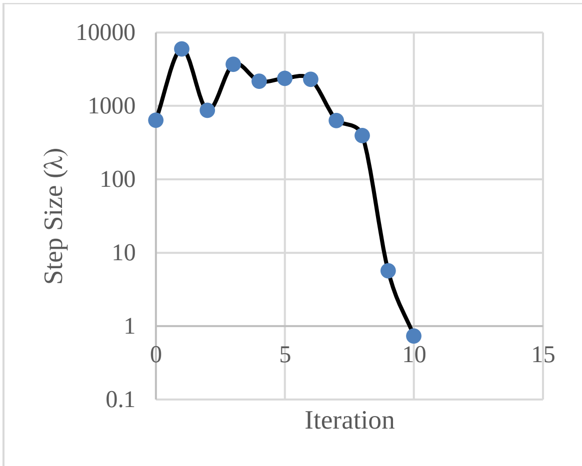

Figure 4. Optimum Stem Size vs Iteration for (5, 25.8, 26.1, 1, NOT PENALIZED)

Finally, the much desire optimal control is shown in Fig. 5 with $u_1 = 6287.1306\ m^3/s$, $u_2 = 368.7722$ m^3/s, $u_3 = 1692.9600\ m^3/s$ and finally $u_4 = 1725.8464\ m^3/s$. This was passed into the system model equation and Fig. 6 shows that the control can move the head from an initial value of 25.8 to the desired value of 26.1 after 10 iterations. The error between the desired $(h(T)$ and computed head t $h(t_f)$ is within 2%.

Computation 2: (5, 25.8, 1, u_{max}=3000 m^3/s)

The optimal control of Fig. 6 required over 6000 m^3/s in the first six hour, this may not be available form KHEPS. Hence a penalty can be imposed on the algorithm such that the maximum control cannot be greater than3000 m^3/s. This is the case considered in Computation 2 and against the Not Penalized Computation 1. In Fig. 7, the optimal control decreases gradually after the second partition of time. The trajectories of head presented in Fig. 8 presents a better rise in the head as compared to the overshoot in Figure 6. Hence, it is better to place a penalty on the maximum control when computing direct optimal control with conjugate gradient method.

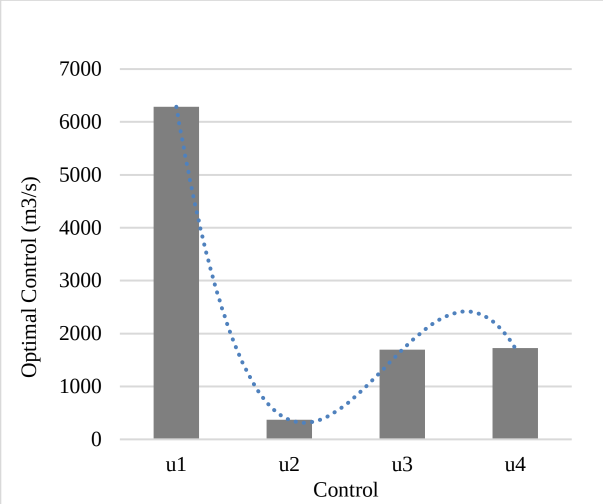

Figure 5. Optimal Control for (5, 25.8, 26.1, 1, NOT PENALIZED)

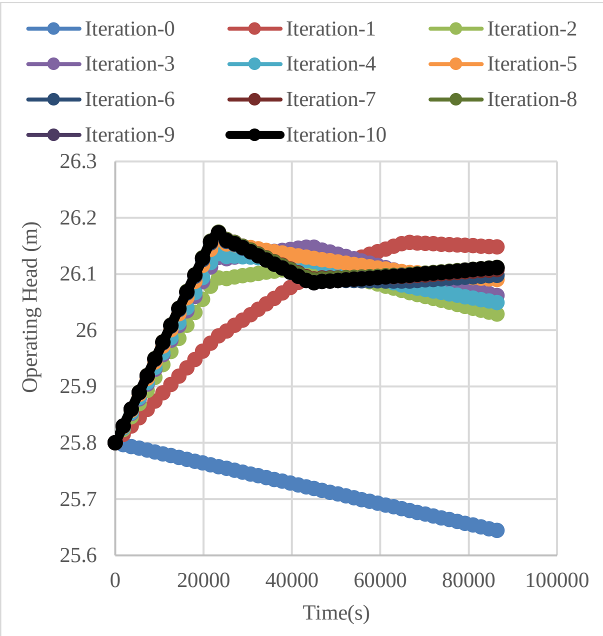

Figure 6. Operating Head Trajectories for (5, 25.8, 26.1, 1, NOT PENALIZED)

Computation 3: (5, 25.5, 2, u_{max}=3000 m^3/s)

Computation 3 presents a situation when the operating head is as low as 25.5m, because of the size of the reservoir was not feasible to raise the operating head to 26.1 m in 24 hours with a maximum inflow of 3000 m^3/s. In such situation, the time can be changes to 48 hours with a partitioning of 12 hours. Fig. 9 shows a gradual decrease in the optimal control such as to achieve the desired trajectory of Figure 10. Control algorithm is robust enough to obtain the optimal control for any desired head from any operating head. The operator will only need to trade the maximum control for duration of time to reach the desired head.

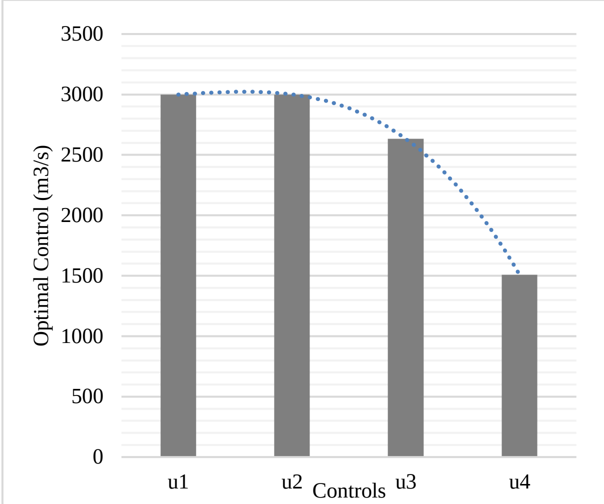

Figure 7. Optimal Control for (5, 25.8, 1, u_{max} = 3000 m^3/s)

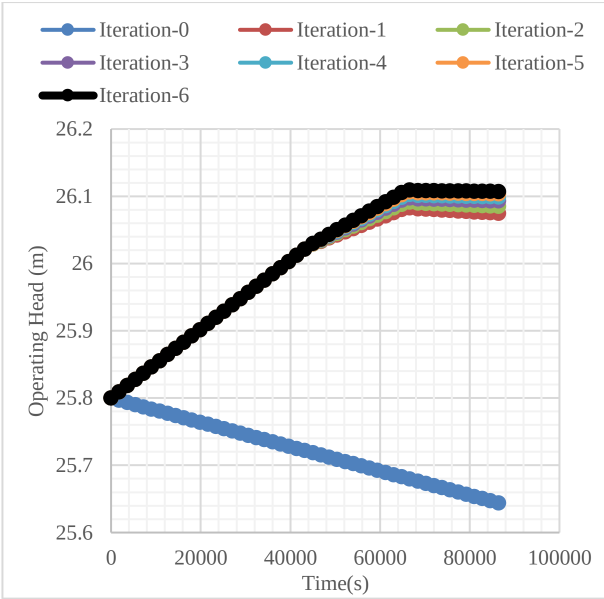

Figure 8. Operating Head Trajectories for (5, 25.8, 1, u_{max} = 3000 m^3/s)

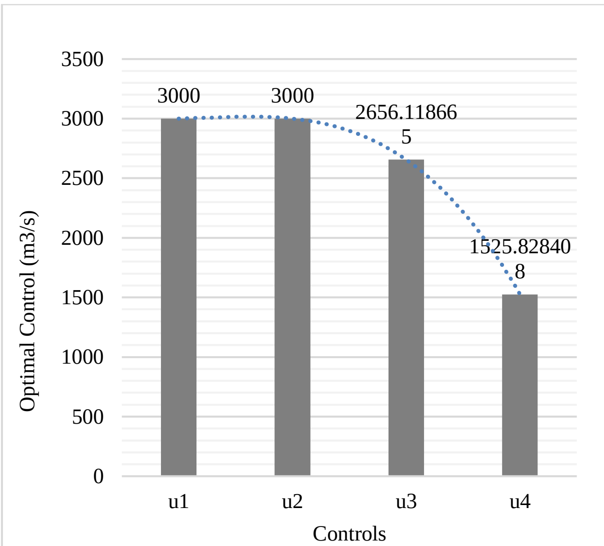

Figure 9. Optimal Control for (5, 25.5, 2, u_{max}=3000 m^3/s)

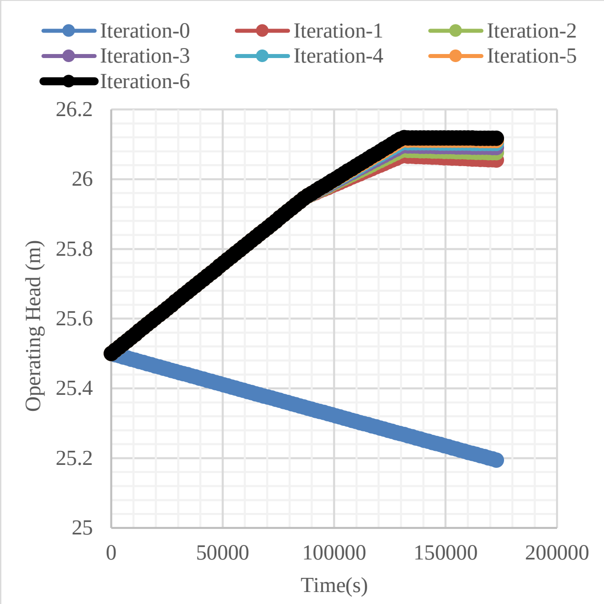

Figure 10. Operating Head Trajectories for (5, 25.5, 2, u_{max}=3000 m^3/s)

Computation 4: (3, 25.8, 1, u_{max}=3000 m^3/s)

The potential of the algorithm was also checked for a condition where the number of operating machines reduces to three. Fig. 11 and 12 presents the optimal control and head trajectory under such condition. A gradual decrease was observed in the optimal control from the first-time interval to the last interval.

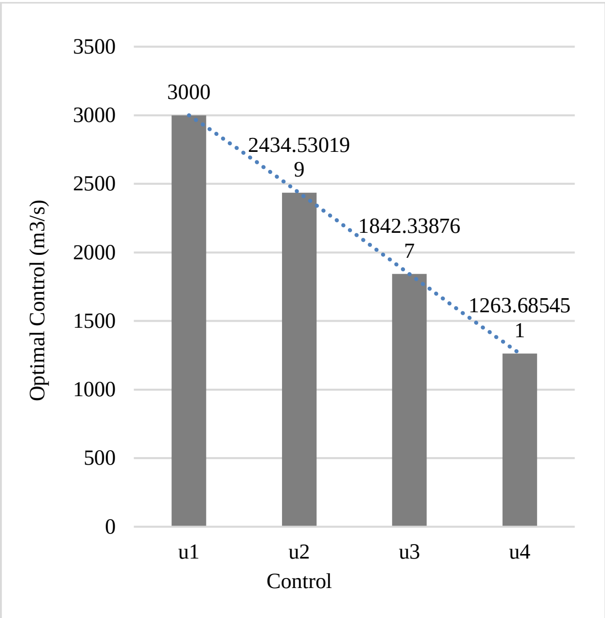

Figure 11. Optimal Control for (3, 25.8, 1, u_{max}=3000 m^3/s)

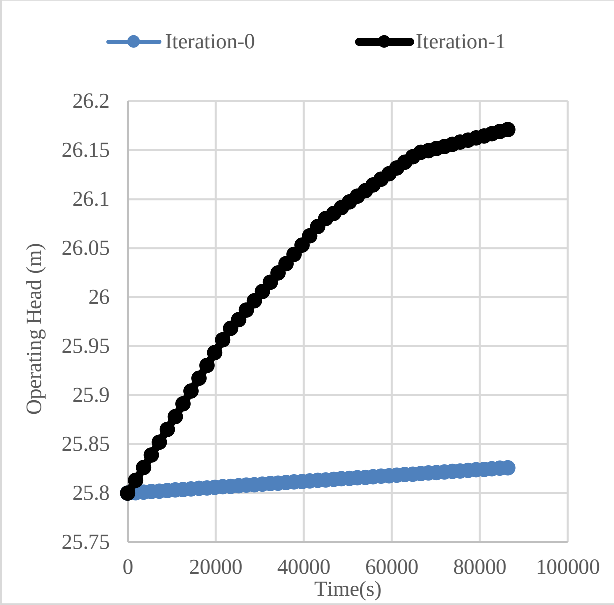

Figure 12. Operating Head Trajectories for (3, 25.8, 1, u_{max}=3000 m^3/s)

Computation 5: (3, 25.8, 1, u_{max}=2500 m^3/s)

From Fig. 7, 9 and 11, it can be observed that the first optimal control equals the penalized value. If the operating machines reduces from 5 units to 3 units, it is also possible to reduce the penalized value to 2500m/s. This is the case presented in Computation 5 and the results are shown in Fig. 13 and 14. The head rises smoothly within 24 hours without unnecessary overshoot and with a smaller number of iterations.

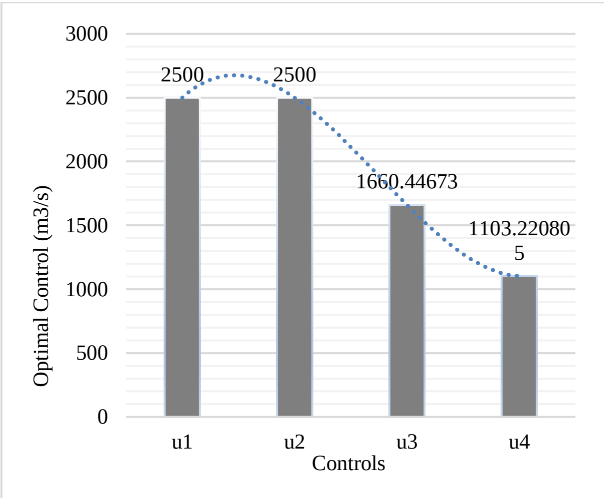

Figure 13. Optimal Control for (3, 25.8, 1, u_{max}=2500 m^3/s)

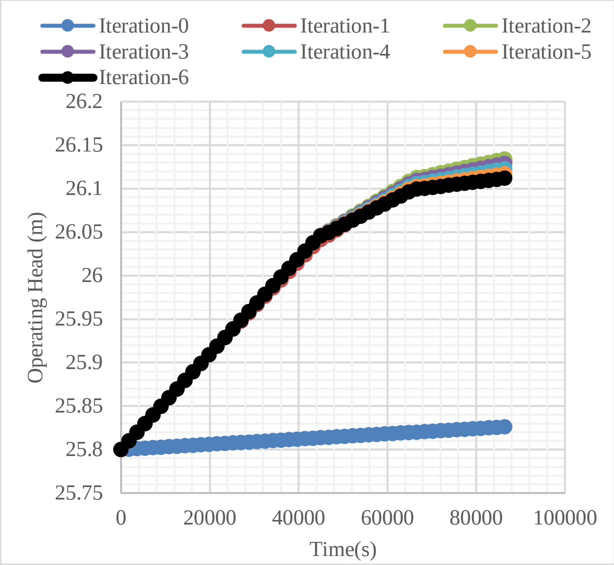

Figure 14: Operating Head Trajectories for (3, 25.8, 1, u_{max}=2500 m^3/s)

3. CONCLUSION

The optimal control of the cascaded Kainji-Jebba hydroelectric power station based on conjugate gradient was successfully carried out. The results show that a direct optimal control of the operating head of JHEPS is possible using conjugate gradient approach. The algorithm is flexible to handle several operating units from 1 to 6 and available control release from KHEPS. Hence, whenever a release is required to raise the head of JHEPS to nominal level, the amount required form KHEPS can be computed for a stipulated number of times. It is also possible for this control algorithm to be running in a real time digital controller such that the head remain constant. The work is recommended for use by operators of the station and support for the development of the physical controller is encouraged.

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